A Class of Analytical Absorbing Boundary Conditions Originating From the Exact Surface Impedance Boundary Condition

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***Abstract*—A new class of analytical absorbing boundary conditions (ABCs) for the truncation of the finite-difference time-domain (FDTD) lattice is introduced. These ABC originate from the exact impedance boundary operator and contain both electric and magnetic fields. The performance of the proposed second-order ABCs is studied with a two-dimensional FDTD program. The results indicate that the proposed ABCs work approximately as well as the third-order analytical ABCs, even if they are essentially as easy to implement as the second-order Mur ABC. Also, in this paper, the relation between the so-called Engquist–Majda operator for the absorption of plane waves and the exact surface impedance boundary condition is discussed.**

I. Introduction

THE use of absorbing boundary conditions (ABCs) to truncate the computational lattice is a very common feature in finite-difference time-domain (FDTD) simulations. Several approaches have been taken during the recent decades to solve this problem. Probably the simplest analytical ABCs are the first- and second-order Mur ABC [1]. They are easy to implement, and the computational burden does not become prohibitive even in large computational domains. The perfectly matched layer (PML) ABC [2] should be used when only a very small reflection is allowed. However, the PML requires quite large computational effort and is clearly more complicated to implement into FDTD programs.

In this paper, we introduce a new class of analytical ABCs, which stems from the exact impedance boundary condition simulating empty half-space. The key characteristic feature of this approach is the presence of both tangential electric and magnetic fields in the ABC, which, as we show, leads to a possibility to derive ABCs with only second-order differentiations, which is approximately as accurate as the conventional third-order schemes.

We begin this paper with a discussion of an interesting connection between the exact impedance boundary condition for isotropic half-space and the Engquist–Majda equation [5] for the absorption of plane waves. The Engquist–Majda wave operator operates on one field component. Thus, all the resulting ABCs are always for only one field component. In Section II, we introduce the exact surface impedance boundary condition (SIBC) for modeling the behavior of an isotropic half-space with material parameters ϵ and μ. In the case when $\epsilon = \epsilon_0$ and $\mu = \mu_0$, the resulting SIBC should work as an ABC. The Engquist–Majda wave operator is also derived in this case.

In Section III, a class of ABCs connecting the tangential-field components is derived using a general form of rational approximation to approximate the involved pseudodifferential operator. The performance of the proposed ABCs is studied and comparisons with some previously introduced analytical ABCs are made with a test two-dimensional (2-D) FDTD program. These results are presented in Section IV.

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II. SIBC and Its Relation to the Engquist–Majda Equation

To set up the scene and introduce the necessary relations, we will now discuss the SIBC and the Engquist–Majda equation since both can be used in derivations of analytical ABCs. To terminate the grid of calculation domain, we should somehow simulate a boundary with unbounded free space. It is known that the Engquist–Majda equation applied to every tangential component of the electric field on the boundary can serve this purpose. On the other hand, theoretically, one can demand that the exact SIBC connecting tangential electric and magnetic fields on the same boundary be satisfied. This condition can be derived, for instance, using the equivalent-circuit theory [3]. The SIBC reads

$$\mathbf{E}_t = \overline{\overline{Z}}_s \cdot \mathbf{n} \times \mathbf{H}_t \tag{1}$$

where the impedance operator is of the form

$$\overline{\overline{Z}}_s = \eta \frac{k\left(\overline{\overline{I}}_t + \dfrac{\nabla_t \nabla_t}{k^2}\right)}{\sqrt{k^2 + \nabla_t^2}}. \tag{2}$$

Index t denotes the tangential-field components, and $\mathbf{n}$ is the unit vector pointing outwards from the dielectric (or vacuum) half-space. To simplify the notation, let the interface be located at $y = 0$ with $\mathbf{n} = \mathbf{u}_y$. Using the phasor notation, (1) and (2) for plane waves $e^{-j(k_x x + k_y y + k_z z)}$ take the form

$$\breve{E}_x \mathbf{u}_x + \breve{E}_z \mathbf{u}_z = \eta \frac{\overline{\overline{I}}_t - \dfrac{\mathbf{k}_t \mathbf{k}_t}{k^2}}{\sqrt{1 - \left(\dfrac{k_x^2 + k_z^2}{k^2}\right)}} \cdot \mathbf{u}_y \times \left(\breve{H}_x \mathbf{u}_x + \breve{H}_z \mathbf{u}_z\right). \tag{3}$$

where $\mathbf{k}_t = k_x\mathbf{u}_x + k_z\mathbf{u}_z$ is the tangential component of the wave vector. After simplifying (3), we obtain the SIBCs for the electric-field components

$$\breve{E}_x = \eta \frac{\breve{H}_z - \frac{1}{k^2}\left(k_x^2\breve{H}_z - k_xk_z\breve{H}_x\right)}{\sqrt{1-\frac{k_x^2+k_z^2}{k^2}}} \quad (4)$$

$$\breve{E}_z = \eta \frac{-\breve{H}_x - \frac{1}{k^2}\left(k_xk_z\breve{H}_z - k_z^2\breve{H}_x\right)}{\sqrt{1-\frac{k_x^2+k_z^2}{k^2}}}. \quad (5)$$

Physically, these conditions simply demand that the electric and magnetic fields are related to each other as in plane waves traveling in an infinite isotropic space, thus, the surface impedance on the truncation boundary equals the wave impedance of free space.

It is clear and it is known (see [4]) that the impedance formulation and the Engquist–Majda equation are related, as they express the same feature of the absence of reflection ("matching"). Let us show how the Engquist–Majda equation can be derived from the SIBC for the tangential electric-field component E_z. Similar derivation may be done for E_x. We make use of the y-component of the Maxwell equation

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (6)$$

in the form

$$-jk_x\breve{H}_z + jk_z\breve{H}_x = -\epsilon_0 j\omega \breve{E}_y \quad (7)$$

and the equation

$$-jk_y\breve{E}_z + jk_z\breve{E}_y = -\mu_0 j\omega \breve{H}_x \quad (8)$$

to obtain an equation for the E_z-component only

$$\left(k\sqrt{1-\frac{k_x^2+k_z^2}{k^2}} + k_y\right)\breve{E}_z = 0. \quad (9)$$

Finally, transforming the k's to partial differential operators ($\partial/\partial t \leftrightarrow D_t$, etc.), we obtain

$$\left(\frac{D_t}{c}\sqrt{1-\frac{D_x^2+D_z^2}{\left(\frac{D_t}{c}\right)^2}} - D_y\right)\breve{E}_z = 0. \quad (10)$$

This is the Engquist–Majda pseudodifferential equation. With different approximations of the square root, a class of analytical ABCs can be derived, as is well known from the literature.

One can observe that, in this derivation, the Maxwell equations have been used once more, which involves differentiations of the fields. The same can be said about the ABCs derived from the one-way wave equation. The wave equation already contains second-order derivatives of the fields. On the other hand, the impedance boundary condition follows directly from the Maxwell equations. As we already pointed out, the SIBC is simply the relation between the electric and magnetic fields in plane waves. This suggests that one can expect to obtain a better accuracy from an ABC if the same approximation for the square root is used in the SIBC as compared to ABCs based on the Engquist–Majda operator or on the one-way wave equation. This possibility will be explored below.

III. Derivation of a Class of Analytical ABCs

In the following derivation, we consider the 2-D TM_z case. Let us first introduce the rational approximation of the function $\sqrt{1-x^2}$ on the interval $-1 \le x \le 1$ in the form

$$\sqrt{1-x^2} \approx \frac{a+bx^2}{1+dx^2}. \quad (11)$$

Using this approximation, (5) takes the form

$$\left(ak^2+bk_x^2\right)\breve{E}_z = -\eta\left(k^2+dk_x^2\right)\breve{H}_x \quad (12)$$

with $\eta = \sqrt{\mu_0/\epsilon_0}$. Using the relation $k^2 = \omega^2/c^2$ with the Fourier-transform pairs $j\omega \leftrightarrow \partial/\partial t$ and $-jk_x \leftrightarrow \partial/\partial x$, we obtain the partial differential equation (PDE)

$$\frac{a}{c^2}\frac{\partial^2 E_z}{\partial t^2} + b\frac{\partial^2 E_z}{\partial x^2} = -\frac{\eta}{c^2}\frac{\partial^2 H_x}{\partial t^2} - \eta d\frac{\partial^2 H_x}{\partial x^2}. \quad (13)$$

It is worth noting that the second-order time derivative of H_x can be expressed using the electric field E_z

$$\frac{\partial^2 H_x}{\partial t^2} = -\frac{c}{\eta}\frac{\partial^2 E_z}{\partial y \partial t}. \quad (14)$$

The resulting equation is

$$\frac{a}{c^2}\frac{\partial^2 E_z}{\partial t^2} + b\frac{\partial^2 E_z}{\partial x^2} = \frac{1}{c}\frac{\partial^2 E_z}{\partial y \partial t} - \eta d\frac{\partial^2 H_x}{\partial x^2}. \quad (15)$$

We discretize this equation about an auxiliary lattice point, located a half-cell away from the interface. Note that we do not have to neglect any spatial or temporal differences. The resulting update equation for the electric field is

$$\begin{aligned}
E_z|_{i,0}^{n+1} &= -E_z|_{i,1}^{n-1} + \frac{2a\Delta y}{c\Delta t + a\Delta y}\left(E_z|_{i,0}^{n} + E_z|_{i,1}^{n}\right) \\
&+ \frac{c\Delta t - a\Delta y}{c\Delta t + a\Delta y}\left(E_z|_{i,1}^{n+1} + E_z|_{i,0}^{n-1}\right) \\
&- \frac{b(c\Delta t)^2\Delta y}{\Delta x^2(c\Delta t + a\Delta y)}\begin{pmatrix} E_z|_{i+1,1}^{n} - 2E_z|_{i,1}^{n} + E_z|_{i-1,1}^{n} + \\ E_z|_{i+1,0}^{n} - 2E_z|_{i,0}^{n} + E_z|_{i-1,0}^{n} \end{pmatrix} \\
&- \frac{\eta d(c\Delta t)^2\Delta y}{\Delta x^2(c\Delta t + a\Delta y)} \\
&\cdot \begin{pmatrix} H_x|_{i+1,1/2}^{n+1/2} - 2H_x|_{i,1/2}^{n+1/2} + H_x|_{i-1,1/2}^{n+1/2} + \\ H_x|_{i+1,1/2}^{n-1/2} - 2H_x|_{i,1/2}^{n-1/2} + H_x|_{i-1,1/2}^{n-1/2} \end{pmatrix}. \quad (16)
\end{aligned}$$

It may be noticed that, in the cases $a = 1$, $b = d = 0$ and $a = 1$, $b = -1/2$, $d = 0$, this new ABC reduces to the first- and second-order Mur ABCs, respectively. It is well known that these ABCs, however, are not quite good because they are based on rather coarse approximations of the square root in (11). By choosing the coefficients of the rational approximation appropriately, we obtain much better ABCs than the second-order Mur ABC, while retaining an essentially similar complexity of the update equation. A table of coefficients corresponding to different approximation methods designed to approximate (11) can be found in [9].

Usually, the third-order ABCs resulting from the rational approximation of the square root in the form of (11) are formulated as third-order PDEs for one field component. We will show in numerical examples that we obtain similar performance by our second-order PDE where both the electric and magnetic fields are present. Our analysis yields similar results as presented by Wang *et al.* in [6] and Ramadan and Niazi in [7] in the 2-D case, but the starting point is different, and our derivations establish the connection between SIBCs and ABCs. We next extend the proposed method to the general three-dimensional (3-D) case.

IV. Analytical ABCs in the 3-D Case

In the 3-D situation, the derivatives in the numerator of (5) do not drop out. Hence, the analytical ABC cannot be directly formulated as a second-order PDE. The reduction of order is, however, possible if we use Maxwell's equation and the following definitions:

$$\begin{aligned}\breve{A}_x &= \frac{k_y}{\eta k}\breve{E}_z\\ \breve{A}_z &= -\frac{k_y}{\eta k}\breve{E}_x.\end{aligned} \tag{17}$$

With these definitions, we have the following equations relating the electric field and the just-defined auxiliary field quantities

$$\begin{aligned}\left(kk_y + ak^2 + b\left(k_x^2+k_z^2\right)\right)\breve{E}_z &= -\eta d\left(k_x^2+k_z^2\right)\breve{A}_x\\ \left(kk_y + ak^2 + b\left(k_x^2+k_z^2\right)\right)\breve{E}_x &= \eta d\left(k_x^2+k_z^2\right)\breve{A}_z.\end{aligned} \tag{18}$$

Transforming these equations into the time domain, we obtain the following PDEs:

$$\begin{aligned}&\frac{\partial^2 E_z}{\partial y \partial t} - \frac{a}{c}\frac{\partial^2 E_z}{\partial t^2} - bc\left(\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial z^2}\right)\\ &\qquad - d\eta c\left(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial z^2}\right) = 0\\ &\frac{\partial^2 E_x}{\partial y \partial t} - \frac{a}{c}\frac{\partial^2 E_x}{\partial t^2} - bc\left(\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial z^2}\right)\\ &\qquad + d\eta c\left(\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial z^2}\right) = 0.\end{aligned} \tag{19}$$

These PDEs can be discretized in a similar way as in two dimensions. The definition of the auxiliary field quantities A_x and A_z does not induce extra complexity to the FDTD algorithm since they are actually already calculated in the normal Yee updating scheme as parts of the magnetic-field components H_x and H_z, respectively.[1] Thus, we just need to save those components for later use in the discretized version of (19). As an example, consider the magnetic field H_x. We may think that we split the H_x as $H_x = A_x + B_x$. We can then update the auxiliary variables A_x and B_x according to

$$\begin{aligned}A_x|_{i,j+1/2,k+1/2}^{n+1/2} &= A_x|_{i,j+1/2,k+1/2}^{n-1/2} - \frac{\Delta t}{\mu_0 \Delta y}\\ &\qquad \cdot\left(E_z|_{i,j+1,k+1/2} - E_z|_{i,j,k+1/2}^{n}\right)\\ B_x|_{i,j+1/2,k+1/2}^{n+1/2} &= B_x|_{i,j+1/2,k+1/2}^{n-1/2} + \frac{\Delta t}{\mu_0 \Delta z}\\ &\qquad \cdot\left(E_y|_{i,j+1/2,k+1} - E_y|_{i,j+1/2,k}^{n}\right).\end{aligned} \tag{20}$$

These update equations are only needed on the boundary. After updating the A's and B's, we add them together to get the magnetic fields on the boundary.

The ABCs in (19) are analogous to the 2-D ABCs in a sense that the second-order Mur ABC can be recovered by suitably choosing the parameters. However, in contrast to the Engquist–Majda conditions, third-order accuracy is expected with these second-order conditions when a better rational approximation is used.

V. Validation of the ABCs With Comparison Studies

To study the performance of the ABC in (17), we have constructed a 2-D test lattice with the size of 20×200 cells. The source is a hard source at the center of the lattice with the time dependence

$$E_z|_{20,100}^{n} = \begin{cases}\frac{1}{32}\Big[10 - 15\cos(2\pi f n\Delta t) + 6\cos(4\pi f n \Delta t)\\ \qquad - \cos(6\pi f n \Delta t)\Big], & n \le 30\\ 0, & n > 30\end{cases} \tag{21}$$

where $f = 1$ GHz, $\Delta t = 0.9999\Delta x/(\sqrt{2}c)$ and $\Delta x = \Delta y = 0.015$ m. This pulse has a very smooth decay to zero. The reflection errors are studied on the left-hand side of the lattice. The local error calculated at time step $n = 150$ is shown in Fig. 1. It is clear that our ABCs corresponding to the Padé approximation ($a = 1,\ b = -0.75,\ d = -0.25$) and the Chebyshev on a subinterval ($a = 0.99973,\ b = -0.80864,\ d = -0.31657$) approximation of the square root in (11) are much better than the second-order Mur ABC. To enable comparisons with a third-order method, the Liao's third-order ABC [8] was implemented. Actually, the performance of the proposed ABC with Chebyshev on a subinterval is seen to be about as good as that of the third-order Liao's ABC. The standard discretization of the third-order analytical PDE produces exactly the same results as the proposed analytical ABCs based on the second-order PDE, thus they have not been plotted in this figure.

Let us next study the global errors, which are probably a better measure for the performance of the ABC, since the squared errors are calculated and spatially summed over the whole 2-D FDTD lattice. The global errors as a function of time steps are shown in Fig. 2. It is evident from this figure that our second-order ABCs perform much better than the second-order Mur's ABC. In the case of Chebyshev on a subinterval, the global error is seen to be even smaller than for Liao's third-order ABC. For time steps from 0 to 70, it is seen that the Padé approximation

[1]The subindexes of A refer to these two magnetic-field components.

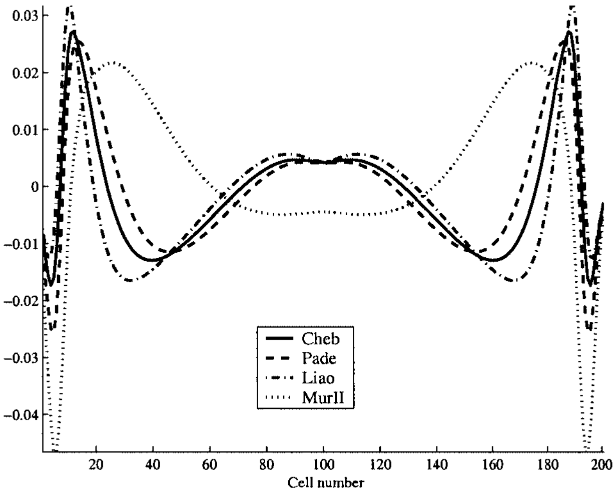

Fig. 1. Local error on one side of the lattice at time step $n = 150$.

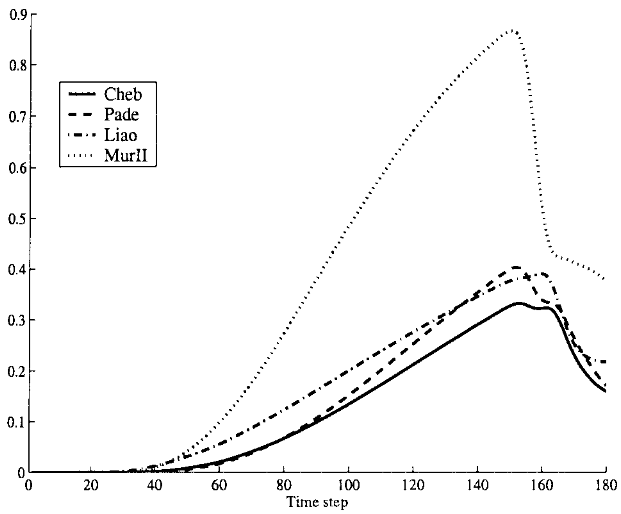

Fig. 2. Global error power reflected back to the lattice as a function of time.

provides the smallest global error. This is natural because the Padé approximation corresponds to having a triple zero of the wave reflection for normal incidence and, at earlier times, the components propagating at grazing angles are small. The decay of the global error after approximately $n = 150$ just reflects the fact that the source has gone to zero some time ago and the errors become smaller.

Comparison of second-order Mur and higher order Lindman ABCs with PML ABCs can be found in [10]. It is evident that PMLs outperform those analytical ABCs. However, the use of analytical ABCs allows the efficient and accurate enough solving of many practical problems without the need to invoke PML ABCs.

VI. Conclusions

A new class of analytical ABCs has been derived and some comparisons have been made with other analytical ABCs. These new ABCs contain both electric and magnetic fields and, physically, they are closely related to the exact SIBC. In the 2-D case, it was found that, by keeping both the tangential electric and magnetic field in the derivation, we may reduce the order of the PDE from 3 to 2 while keeping the performance of the third-order ABC. In the 3-D case, we may reduce the order of the PDE by introducing two auxiliary variables that can be conveniently updated in the standard Yee algorithm. Also, the connection between an exact SIBC and the Engquist–Majda analytical ABC has been discussed for better understanding of the background of the new method.

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